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Symmetry and chaos in the complex Ginzburg±Landau equationÐ I. Re¯ ectional symmetries

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Abstract. The complex Ginzburg±Landau (CGL) equation on a one-dimensional domain with periodic boundary conditions has a number of diþerent symmetries. Solutions of the CGL equation may or may not be ®xed by the action of these symmetries. We investigate the stability of chaotic solutions with some re¯ ectional symmetry to perturbations which break that symmetry. This can be achieved by considering the isotypic decomposition of the space and ®nding the dominant Lyapunov exponent associated with each isotypic component. Our numerical results indicate that for most parameter values, chaotic solutions that have been restricted to lie in invariant subspaces are unstable to perturbations out of these subspaces, leading us to conclude that for these parameter values arbitrary initial conditions will generically evolve to a solution with the minimum amount of symmetry allowable. We have also found a small region of parameter space in which chaotic solutions that are even are stable with respect to odd perturbations.

1 Introduction

Pattern form

space dimension so that the solutions of (1) exhibit soft turbulence (Bartuccelli et al., 1990).

The CGL equation (1) is equivariant with respect to various symmetry groups and its solutions (chaotic or otherwise) may possess one or more of these symmetries. The aim of this work is to numerically investigate solutions of the CGL equatio

The space X can be decomposed as a direct sum of irreducible subspaces

$$X = \sum_i \oplus V_i$$

If we g

3 Lyapunov exponents and symmetry

The way that symmetry aþects the determination of Lyapunov exponents was considered in Aston and Dellnitz (1995) and applied to systems of coupled oscillators. We brie¯ y review the be found using the vector form of the variational equation restricted to W_k given by

$$\dot{u}_k = D_k f(x(t)) u_k, \qquad u_k(0) = w_k \in W_k \tag{8}$$

and is g

Thus, any chaotic attractor which is not spatially uniform has three zero Lyapunov exponents associated with it. If the attractor lies in Fix(Σ) for some subgroup Σ of Γ, then the zero Lyapunov exponents may occur in different Σ-isotypic components. These can be determined by finding which isotypic components contain the trajectories $LA(x, t)$ for each $L \in$, . However, for any symmetry, $L_h A(x, t)$ and $L_b A(x, t)$ will always have the same symmetry as the solution trajectory and so occur in $W_1 = \mathrm{Fix}(\Sigma)$ whereas in some cases, $L_a A(x, t)$ may occur in a different isotypic component.

As a simple example, consider the CGL equation with homogeneous Neumann boundary conditions which is equivalent to considering solutions which are invariant under the reflectional symmetry s_1. Thus, the symmetries of the solution are given by $\Sigma = \{I, s_1\} \simeq Z_2$. The Σ-isotypic components are $W_1 = \mathrm{Fix}(\Sigma)$ which consists of all even periodic functions and W_2 which consists of all odd periodic functions. When only a reflectional symmetry is involved, the isotypic components W_1 and W_2 are often referred to as the symmetric and antisymmetric spaces, respect(c)]TJ10012(n)-20m

$\simeq Z_2 \times Z_2$

We note that

$$r_p s_1 A(x, t) = A(p - x, t), \qquad r_p s_2 A(x, t) = - A(x + p, t)$$

Thus, functions ®xed by $s_1 s_2$ are odd, functions ®xed by $r_p s_1$ are even about $p/2$ and functions ®xed by $r_p s_2$ satisfy $A(x + p, t) = - A(x, t)$. Clearly if two of these are satis®ed then so is the third. Thus $\mathrm{Fix}(\Sigma_3)$ consists of functions which are odd (about zero)

$$A(x,t) \in W_3 \Rightarrow A(x,t) = \sum_{k=1}^{\infty} b_k(t) \sin 2kx + i\left(\sum_{k=1}^{\infty} c_k(t) \sin 2kx\right)$$

$$A(x,t) \in W_4 \Rightarrow A(x,t) = \frac{b_0(t)}{2} + \sum_{k=1}^{\infty} b_k(t) \cos 2kx + i\left(\frac{c_0(t)}{2} + \sum_{k=1}^{\infty} c_k(t) \cos 2kx\right)$$

5 Numerical method

In this section we briefly describe the numerical method used for calculating the solutions shown in Section 6. We use a pseudo-spectral method coded in Matlab with time integration performed by a variable step-size Runge–Kutta method. We show details for only the real part of $A(x,t)$; the imaginary part is dealt with similarly. We write the real part of $A(x,t)$ at the points $\{x_n\}$ as

$$A_r(x_n, t) = \frac{1}{N} \sum_{k=0}^{N-1} X_k(t) \exp(ikx_n), \qquad 1 \leqslant n \leqslant N \tag{9}$$

where

$$x_n = \left(\frac{n-1}{N}\right) 2\pi$$

(This is just the inverse discrete Fourier transform of $\{X_k(t)\}$.) The spectral coeý (

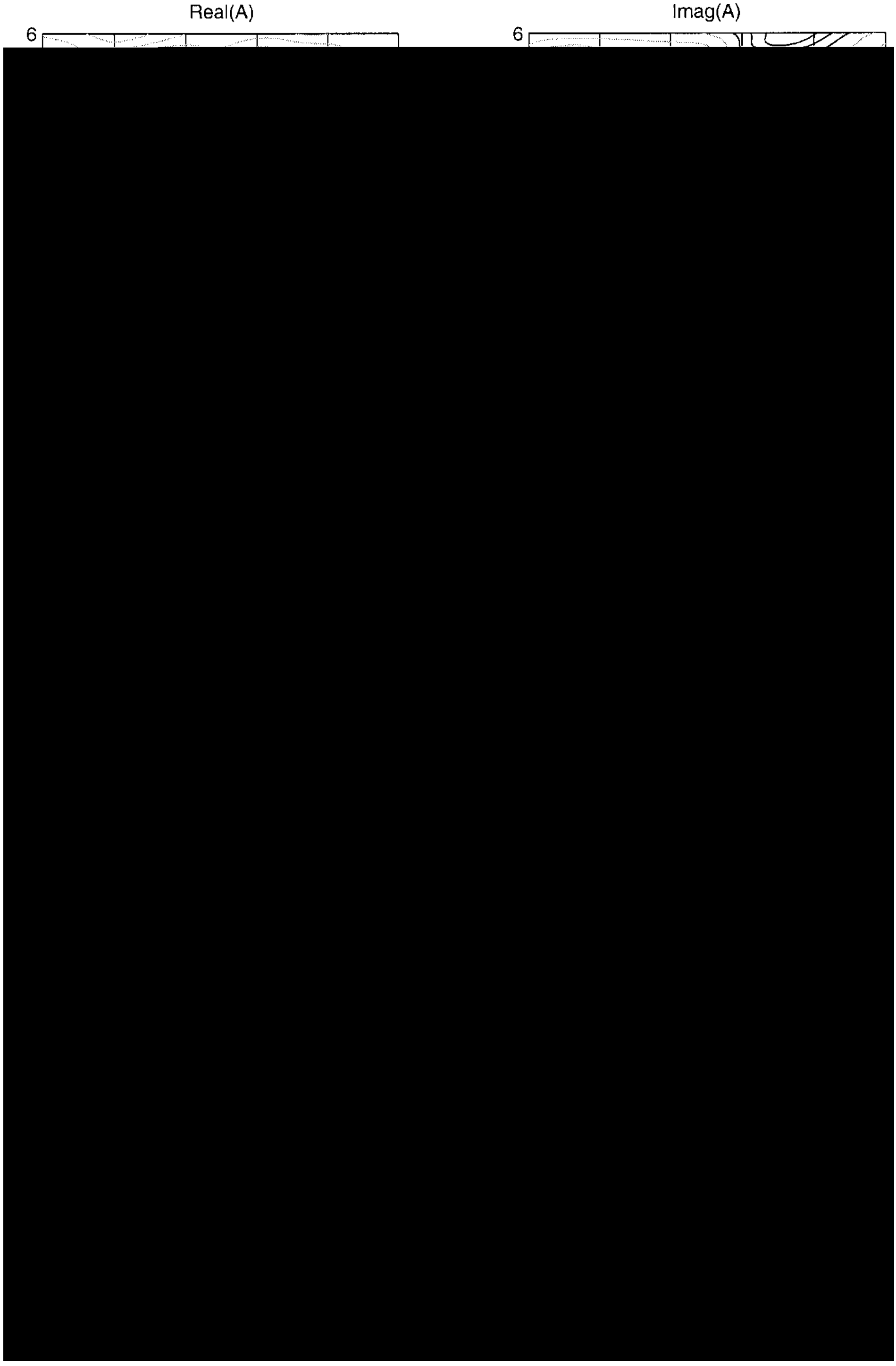

Fig. 1. A periodic even solution that is unstable to odd perturbations, leading to non-symmetric chaos. Parameter values: $R = 16$, $l = 1$, $m = -7$. A small odd perturbation is introduced at $t = 2$.

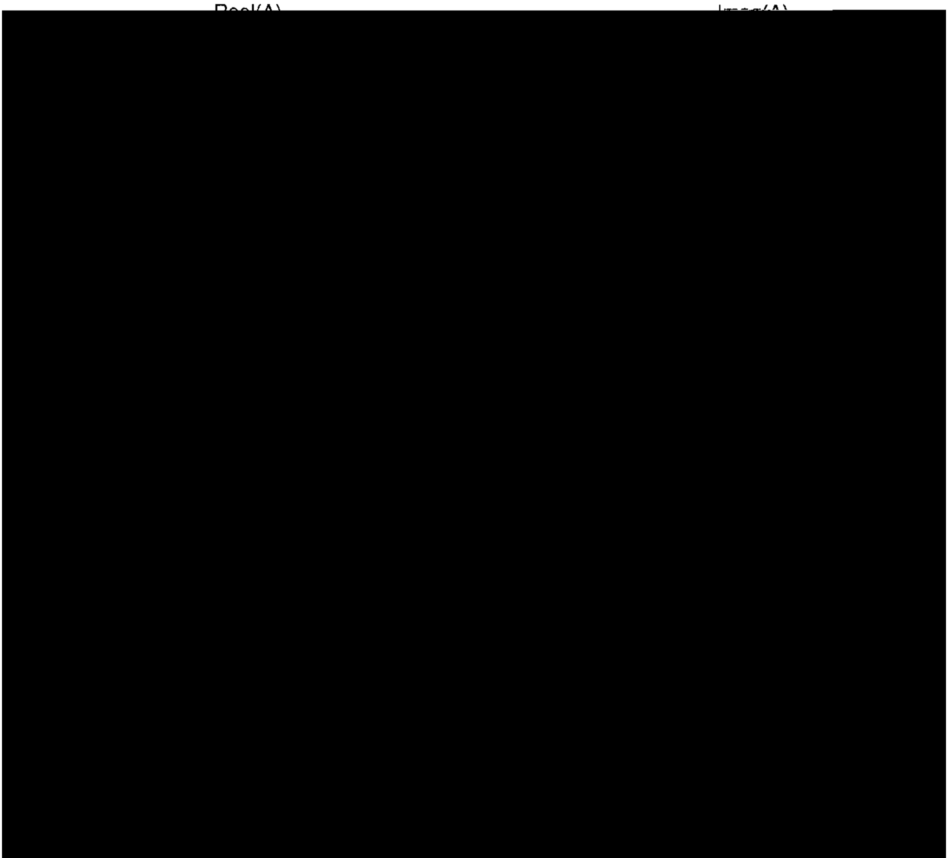

Fig. 2. A chaotic even solution that is unstable to odd perturbations, leading to non-symmetric chaos. Parameter values: $R = 16$, $l = 5$, $m = -7$. A small odd perturbation is introduced at $t = 2$.

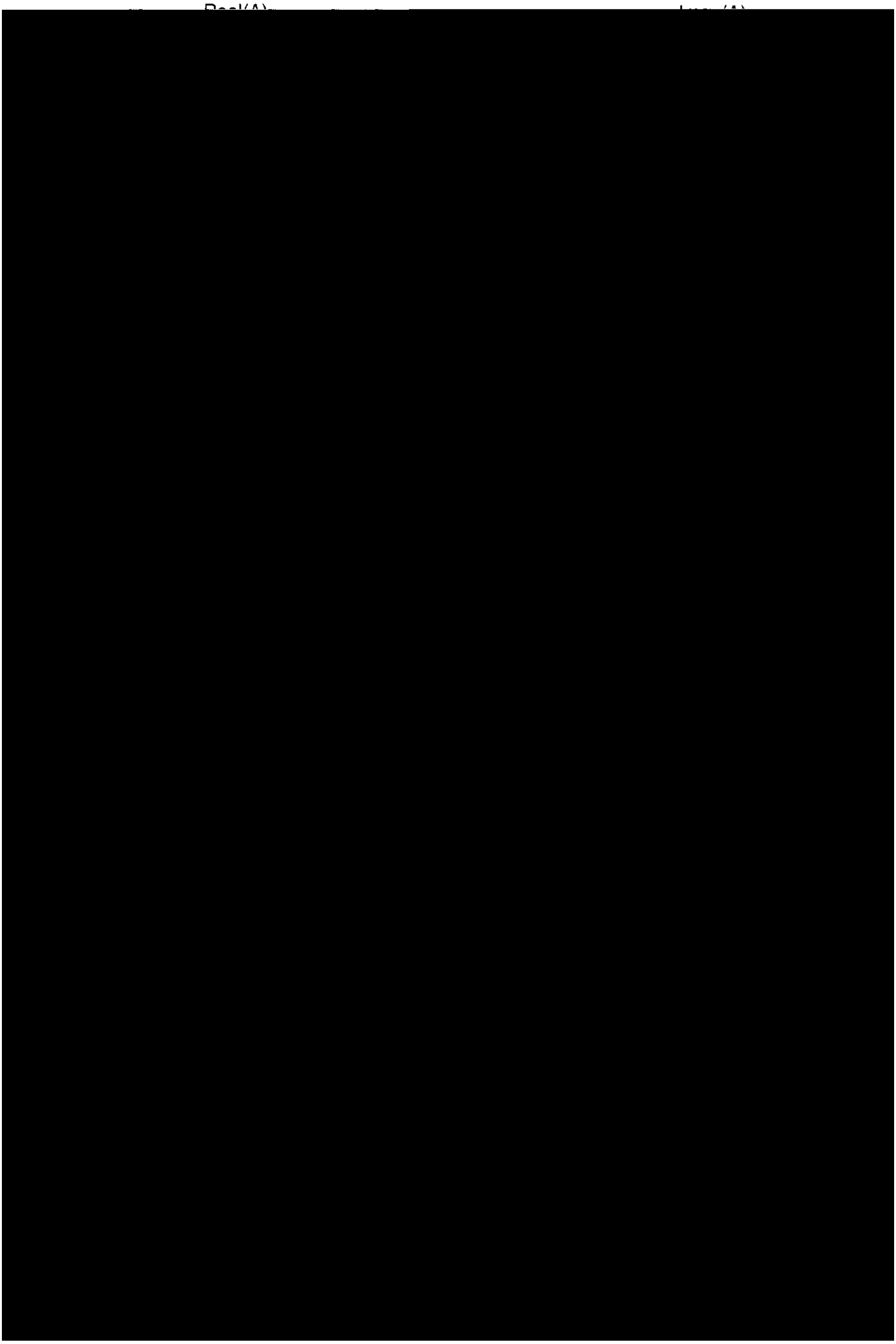

Fig. 3. A chaotic odd solution that is unstable to even perturbations. Parameter

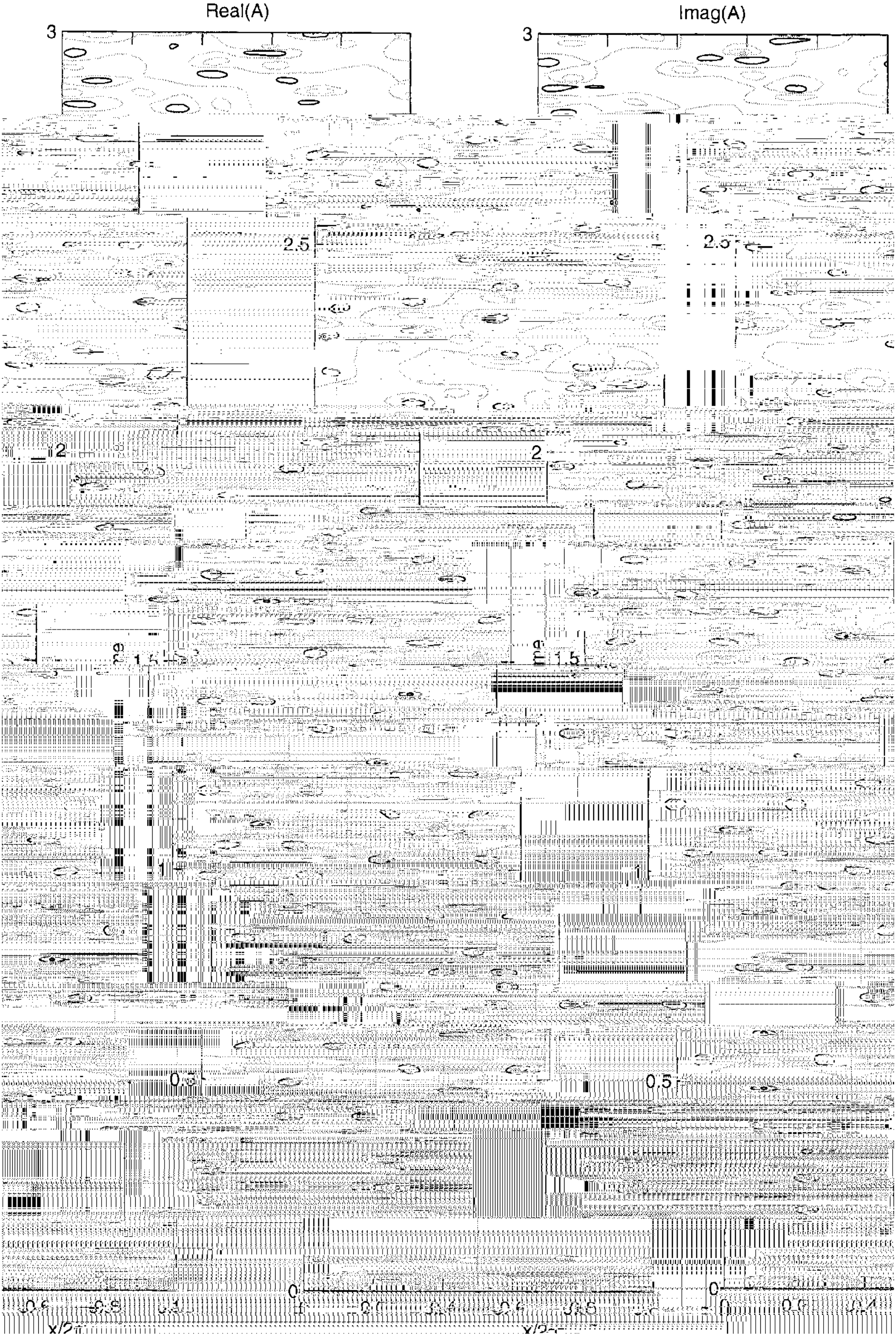

Fig. 4. A chaotic solution that is odd about 0 and even about $p/2$. Parameter values: $R = 16$, $m = -5$, $l = 12$. A small perturbation that is neither odd about 0 nor even about $p/2$ has been added at $t = 1$.

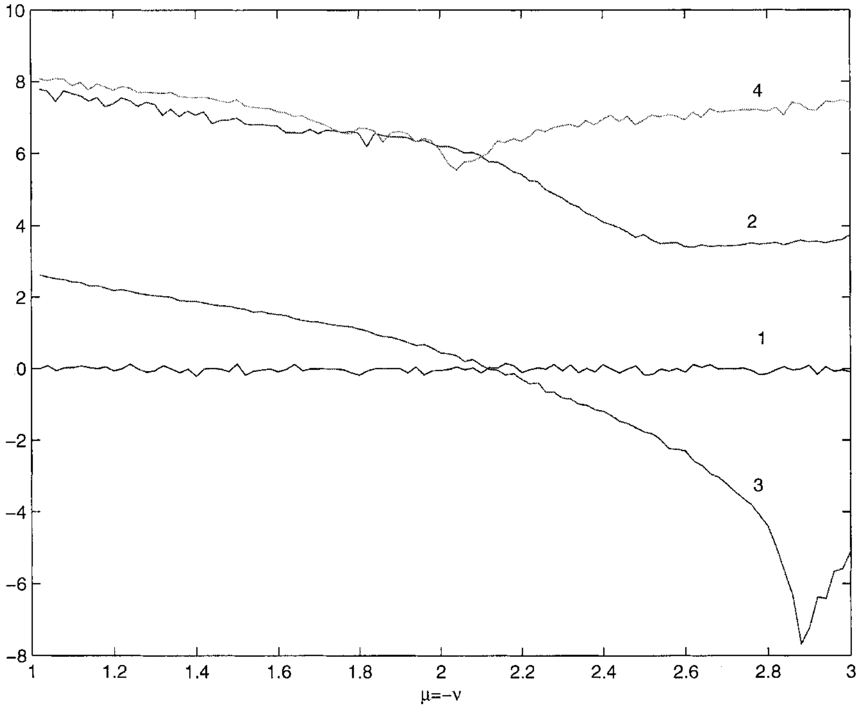

Fig. 5. Dominant Lyapunov exponents for a solution in Fix(Σ_3) with $R = 16$. 1: perturbations in W_1 (= Fix(Σ_3)), 2: perturbations in W_2, 3: perturbations in W_3, 4: perturbations in W_4.

with the signs of the corresponding Lyapunov exponents shown in Fig. 5. Note that the Lyapunov exponent associated with perturbations in W_4 is larger than for perturbations in W_2 and also that the solution component in W_1 ceases to be periodic when the W_4 component becomes similar in magnitude to it. This is the point at

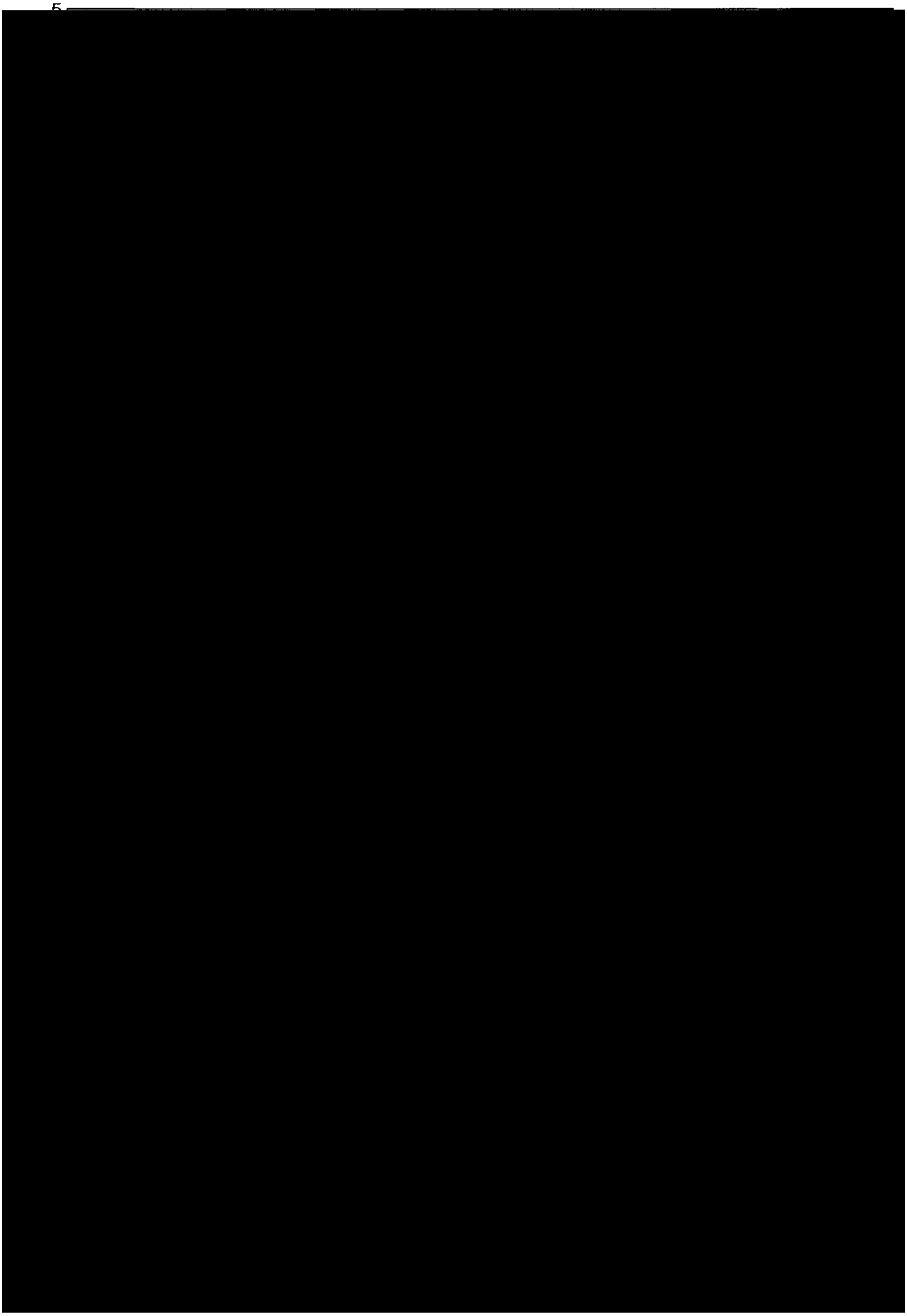

Fig. 6. Growth and decay of functions A_2, A_4 and A_3 over time with $R = 16$, $l = -m = 2.9$. In each case a small perturbation has been added at $t = 1.5$ to a solution in $\mathrm{Fix}(\Sigma_3)$.

although we added a small odd perturbation at time $t = 2s$ to demonstrate instability in this isotypic component, strictly speaking this was not necessary as, given suý cient time, the numerical errors introduced by the Fourier transform

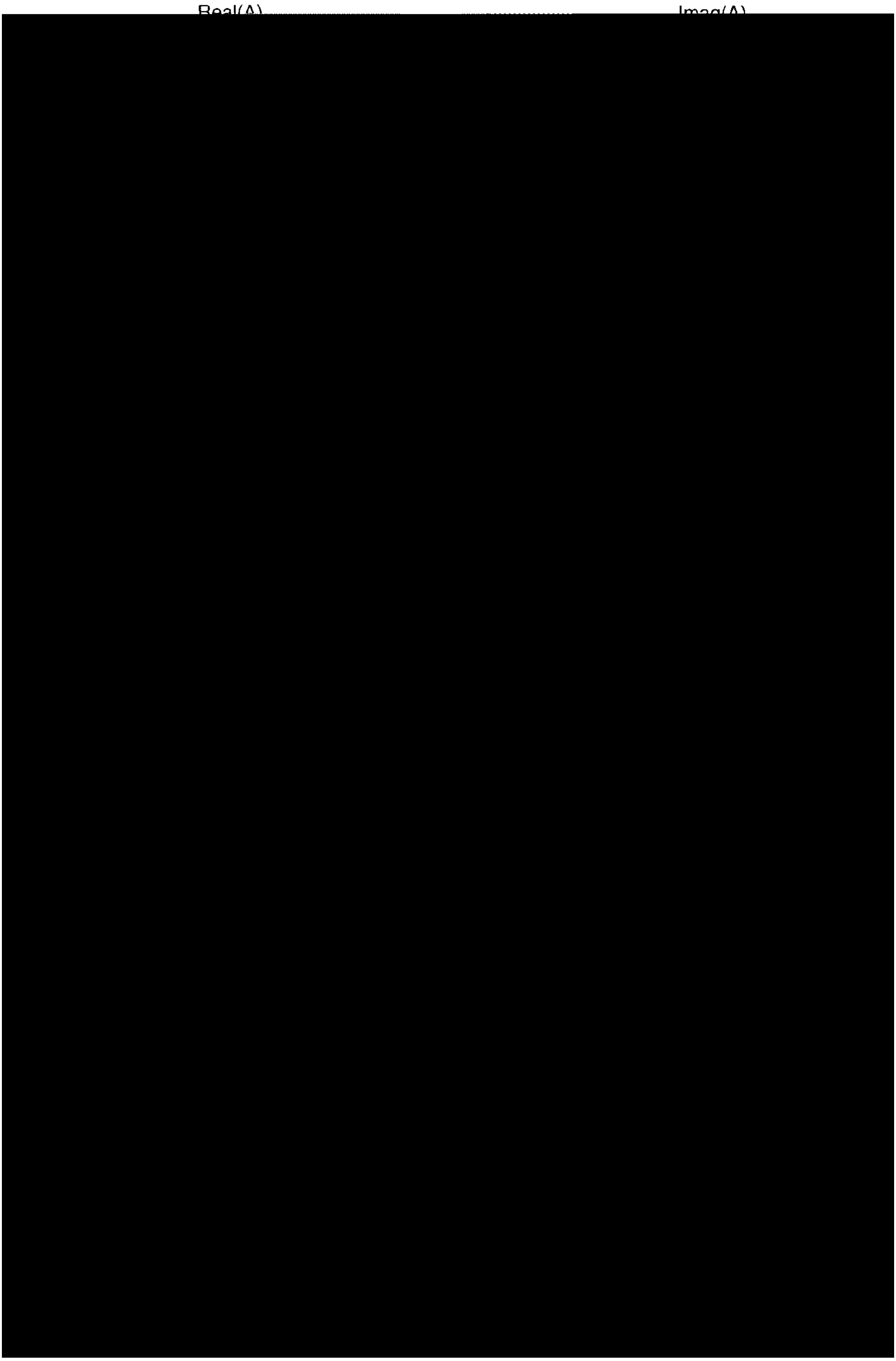

Fig. 8. A stable chaotic solution that is even about the origin. Parameter values: $R = 1.05$, $m = 4$, $l = -4$.

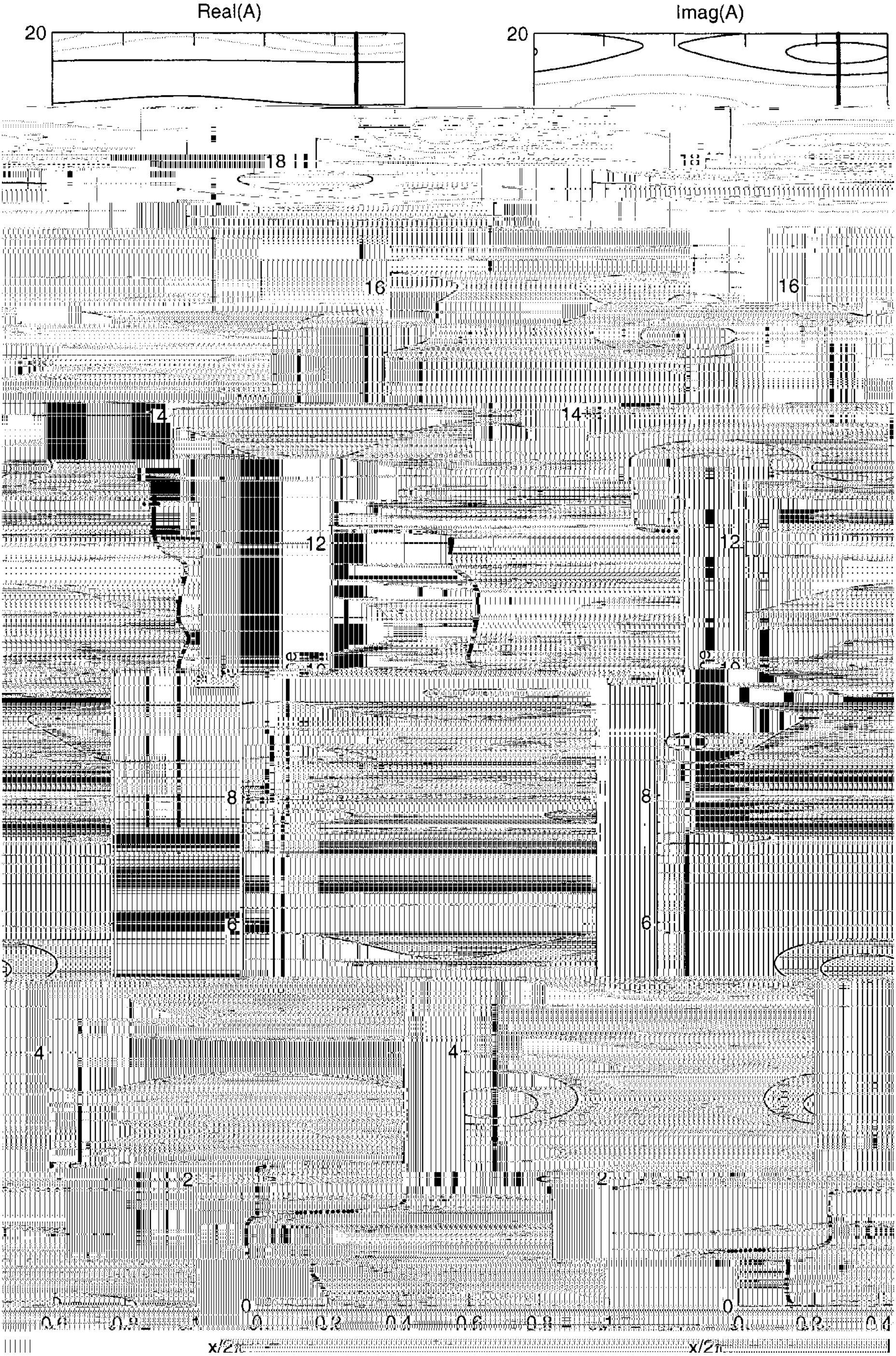

Fig. 9. Demonstration of the orbital stability of an even chaotic solution. The dark line is one of the two points about which the ®rst mode is even, for both the real

combination of $\sin x$ and $\cos x$) is even, for both the real and imaginary parts of A. We see that by time $t = 3$ s the solution has settled to being even about an x value of approximately $0.1 \times 2p$. A large randomly chosen perturbation was added at $t = 10$ and we see that the solution quickly settled to being even about an x value of approximately $0.87 \times 2p$. Note that during the transients the ®rst modes of the real and imaginary parts of A are even about diþerent points, but on the attractor they are even about the same point, as they must be for an even solution. See Golubitsky *et al.* (1988) for more details on orbital stability.

Despite searching, we could not ®nd any evidence of a `blowout' bifurcation (Ashwin *et al.*, 1998) in which the even solution remains chaotic while the dominant Lyapunov exponent in the normal direction changes from zero to positive as a parameter is varied. The reason for this is that, as shown in Fig. 7, the solution in the even subspace becomes periodic or quasiperiodic before the normal Lyapunov exponent becomes positive.

7 Conclusions

Our numerical results show that for much of the parameter space for the C