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A homoclinic hierarchy

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Abstract

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A homoclinic orbit of an autonomous ordinary [redacted] and $x_H(0) \neq x_0$. In typical (e.g. non-Hamiltonian) [redacted] $\epsilon)\setminus\{\mu_H\}$ there is no homoclinic orbit close to the [redacted]

orbit which exists at $\mu = \mu_H$. If this is the case we [redacted] recent work has been stimulated by a series of papers [redacted] cations, for which the homoclinic orbit loops several [redacted]

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on the linearized flow near the stationary point. Suppose that the stationary point is hyperbolic. Then, after a change of coordinates we may assume that it is at the origin for all values of μ which are of interest and the family of differential equations can be written in the form

$$\dot{x} = Ax + F(x, \mu) \qquad (1)$$

for $x \in \mathbb{R}^n$, $n \geqslant 2$. Here $F(0, \mu) = 0$, A is a constant $n \times n$ matrix and F is smooth and contains only nonlinear terms. Assume that if $\mu = 0$ then the [redacted] no homoclinic orbits close to x_H (by close we mean [redacted] all $t \in (-\infty, \infty)$). [redacted] and $\{\nu_i\}$, $i = 1, \ldots, n_s$, $n_s + n_u = n$, such that $\mathrm{Re}(\lambda_i)$ [redacted]

$$< \mathrm{Re}(\lambda_1) \leqslant \mathrm{Re}(\lambda_2) \leqslant \ldots \leqslant \mathrm{Re}(\lambda_{n_u}).$$

[redacted]

Typically, trajectories which tend to $x = 0$ as $t \to \infty$ do so tangential to the eigenspace corresponding to those eigenvalues with $\mathrm{Re}(\nu_j) = \mathrm{Re}(\nu_1)$, which we refer to as the dominant stable eigenvalues. Simi- [redacted] those with $\mathrm{Re}(\lambda_j) = \mathrm{Re}(\lambda_1)$. We assume that the homoclinic orbit $x_H(t)$ is typical in this sense.

[redacted] inant eigenvalues is $\{\nu_1, \lambda_1\}$, with $\nu_1, \lambda_1 \in \mathbb{R}$, and $\nu_1 + \lambda_1 \neq 0$.

In this case (which can occur for $n \geqslant 2$), provide some genericity conditions are satisfied, the homo- [redacted] exists in either $\mu < 0$ or $\mu > 0$ [2]. As μ tends to zero from the appropriate side the periodic orbit [redacted] stable if $\nu_1 + \lambda_1 < 0$, otherwise it is a saddle.

(II) *Saddle-focus homoclinic orbit.* The set of dominant eigenvalues is $\{\nu_2, \nu_1, \lambda_1\}$, with $\nu_1 = \nu_2^*$ $\in \mathbb{C} \setminus \mathbb{R}$, $\lambda_1 \in \mathbb{R}$, and $\mathrm{Re}(\nu_1) + \lambda_1 \neq 0$.

This case can occur if $n \geqslant 3$. There are two subcases.

(IIa) $\mathrm{Re}(\nu_1) + \lambda_1 < 0$. The bifurcation is essentially the same as case (I).

(IIb) $\mathrm{Re}(\nu_1) + \lambda_1 > 0$. If $\mu = 0$ there are chaotic solutions in a tubular neighbourhood of the homoclinic orbit. There are sequences of saddle-node bifurcations accumulating on $\mu = 0$ from both sides, and sequences of (geometrically more complicated) [redacted]

(III) *Bifocal homoclinic orbit.* The set of domi- [redacted] $\mathbb{C} \setminus \mathbb{R}$ and $\lambda_1 = \lambda_2^* \in \mathbb{C} \setminus \mathbb{R}$. [redacted] there are more complicated homoclinic bifurcations [redacted]

The results sketched above form the basis of [redacted] about the saddle-node, period-doubling and Hopf [redacted] literature it is extraordinary that (to the best of our knowledge) no unambiguous examples of case (III) have been described to date. There are examples with homoclinic orbits to stationary points satisfying [redacted] [11,12]. A piecewise linear example of case III is described in Ref. [13], and here we use the same [redacted] evidence for the existence of a bifocal homoclinic orbit. In so doing we derive a hierarchy of equations in two, then three, and then four dimensions. Each equation is obtained from the previous system by [redacted] dimension. In principle this construction could be extended to obtain a hierarchy of equations in higher [redacted]

Simple examples of interesting dynamical phenomena have been constructed using a variety of techniques. Arnéodo, Coullet and Tresser [14] used [redacted] the adjoint eigenvectors of the linear part of a "seed" [redacted] and an extra variable in such a way that the linear part of the new equation has the desired spectral condition. We then appeal to perturbation theory and numerical experiment to suggest that the dynami- [redacted] of a homoclinic orbit) is inherited by the new equa- [redacted] can in turn be treated as a "seed" equation and the process can be repeated. The use of adjoint eigenvectors is not entirely necessary (one could try trial and error) but ensures that complete control of the spec- [redacted] are easy to find, so let

$$\dot{x} = Ax + f(x, \mu) \tag{3}$$

[redacted] tion of the plane to itself which contains only nonlinear terms, $f(0, \mu) = 0$ and there is a homoclinic [redacted] $< \lambda_1$ and

[redacted] eigenvectors (see e.g. Ref. [16] for a discussion of adjoint eigenvectors in dynamical systems). Thus $A^T e_s^\dagger = \nu_1 e_s^\dagger$, $A^T e_u^\dagger = \lambda_1 e_u^\dagger$, $e_s^\dagger \cdot e_u = e_u^\dagger \cdot e_s = 0$ and the eigenvectors can be normalized so that $e_s^\dagger \cdot e_s = e_u^\dagger \cdot e_u = 1$.

Eq. (3) is the first member of the homoclinic hierarchy. Now define the extended system

$$\dot{x} = Ax - z e_s + f(x, \mu),$$

$$\dot{z} = \epsilon_1 (e_s^\dagger \cdot x) + \nu_1 z. \tag{5}$$

In coordinates (x_u, x_s, z) defined by $x = x_s e_s +$ [redacted]

$$\dot{x}_u = \lambda_1 x_u, \quad \dot{x}_s = \nu_1 x_s - z, \quad \dot{z} = \epsilon_1 x_s + \nu_1 z, \tag{6}$$

[redacted] with eigenvalues $\lambda_1 > 0$ and $\nu_1 \pm \sqrt{-\epsilon_1}$. Hence if $\epsilon_1 > 0$ the linear part of (3) satisfies the conditions of case (IIa). Since homoclinic bifurcations are typically of codimension one we expect (at least for [redacted] bifurcations in (μ, ϵ_1) parameter space of the form [redacted] then (5) provides an example of case (IIa).

Similarly, if we consider

$$\dot{w} = \epsilon_2 (e_u^\dagger \cdot x) + \lambda_1 w,$$

$$\dot{x} = Ax - w e_u + f(x, \mu), \tag{7}$$

[redacted] and $\lambda_1 \pm \sqrt{-\epsilon_2}$ and so, using (4), under similar assumptions we obtain homoclinic bifurcations of [redacted]

$$\dot{w} = \epsilon_2 (e_u^\dagger \cdot x) + \lambda_1 w,$$

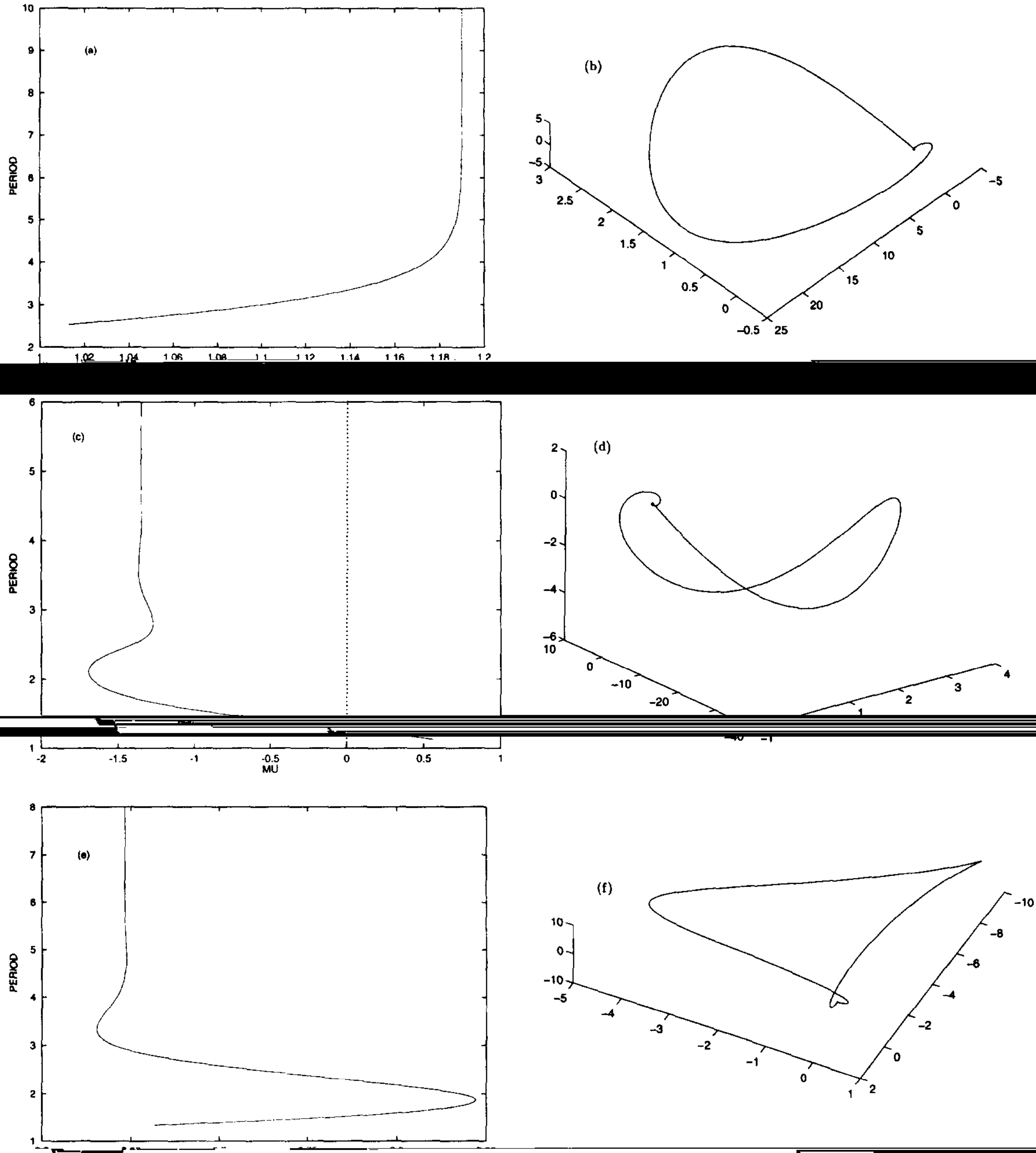
[redacted]

we should be able to find bifocal homoclinic bifurca- [redacted] dimensional system

$$\dot{x} = y, \quad \dot{y} = 6x - y - 6x^2 + \mu xy, \tag{9}$$

for which there is strong numerical evidence that a homoclinic orbit exists if $\mu = \mu_H \approx 1.164371$. For this example, in the notation of (3),

$$A = \begin{pmatrix} 0 & 1 \\ 6 & -1 \end{pmatrix}, \quad f(x, \mu) = \begin{pmatrix} 0 \\ -6x^2 + \mu xy \end{pmatrix}, \tag{10}$$



[redacted]
(μ) for Eq. (5) with $\epsilon_1 = 0.1$ showing the approach of the simple periodic orbit to the homoclinic orbit. (b) A homoclinic orbit of (5) with
[redacted]

so $\lambda = 2$, $\nu = -3$ and (4) is satisfied. A simple [redacted]

$$\dot{w} = \epsilon_2(3x + y) + 2w, \quad \dot{x} = y - \tfrac{1}{5}z - \tfrac{1}{5}w,$$

$$\dot{y} = 6x - y + \tfrac{3}{5}z - \tfrac{2}{5}w - 6x^2 + \mu xy,$$

[redacted]

Although our argument for the existence of homoclinic orbits in (12) (and hence (5) and (7)) is [redacted] $|\epsilon_i|$ $(i = 1, 2)$. We use larger values of the parame- [redacted] features of the orbits, in particular the spiralling motion near the stationary point, is much clearer at these values. In all cases, the approximate parameter [redacted] high period with changing parameter. The homoclinic orbit can be thought of as the limit of this orbit as the period tends to infinity.

Fig. 1 shows the results of three sets of numerical experiments obtained using AUTO [17]. In Figs. 1a, 1b we have set $\epsilon_2 = w = 0$ (equivalent to choosing (5) with A and f given by (10) and the adjoint [redacted] illustrating the familiar logarithmic increase in period as the orbit approaches the homoclinic orbit in case (IIa) with $\epsilon_1 = 0.1$. In Fig. 1b we show a homoclinic orbit for this system with $\epsilon_1 = 16$ and [redacted]

Figs. 1c, 1d shows similar plots for $\epsilon_1 = z = 0$ and $\epsilon_2 = 16$ (equivalent to (7); $z = 0$ is an invariant manifold). In this case, as expected for (IIb), the periodic orbit undergoes a sequence of saddle-node bifurcations as its period tends to infinity. The homoclinic orbit at $\mu \approx -1.351357$ is illustrated in Fig. 1d.

Finally, Figs. 1e, 1f show the analogous pictures [redacted]

the periodic orbit to a bifocal homoclinic orbit [redacted] homoclinic orbit.

[redacted] follow a periodic orbit to very high period provides very strong evidence for the existence of the homoclinic orbit, but we have also done further numerical experiments to add more weight to our claim. The local stable manifold of the origin is tangential to the plane spanned by $e_1 = (0, 0, 0, 1)^T$ and e_s (extended [redacted] manifold is tangential to the plane spanned by e_u (extended to $\mathbb{R}^4$) and $e_4 = (1, 0, 0, 0)^T$. If a homo- [redacted] that a point of intersection lies in the hyperplane [redacted] of this intersection we integrated points on a circle of initial conditions enclosing the origin on the linear approximation to the local unstable manifold for- [redacted] $2 < x < 2.5$ (if such an intersection exists). In this way we obtained a series of points on a curved line segment, U. A similar exercise in reverse time using initial conditions on the linear approximation to the local stable manifold provided a second curved line segment, S. This numerical experiment was repeated at different values of μ. Using polynomial interpola- [redacted] was calculated using Newton's method on the parametrized curves. Now let $\boldsymbol{n}$ be the vector obtained in this way with $\mu = 0.64$, and $\boldsymbol{u}(\mu)$ the vector obtained at nearby values of μ. These results [redacted] $\mathrm{sign}(\boldsymbol{n} \cdot \boldsymbol{u}(\mu))|\boldsymbol{u}(\mu)|$.

A zero of this signed distance function thus indicates an intersection between S and U, and hence the existence of a homoclinic orbit. If, in addition, the sign of the signed distance function changes, then the family of differential equations parametrized by μ passes transversely through the codimension one surface of systems with homoclinic orbits. We found, [redacted] tions and numerically obtained normalized eigenvectors, that for $0.55 < \mu < 0.64$ the signed distance function is positive (and equal to 0.004617 at $\mu = 0.64$ whilst for $0.65 < \mu < 0.71$ the signed distance function is negative (and equal to -0.002365 at $\mu = 0.65$). This strongly suggests that for some values of μ between 0.64 and 0.65 there is a zero of the distance function, and hence a homoclinic orbit for the differential equation (12). Linear interpolation [illegible] cation, in excellent agreement with the value obtained by following periodic orbits.

We have written down a hierarchy of differential equations which illustrate the four fundamental homoclinic bifurcations. In particular, we have obtained a smooth example of a bifocal homoclinic bifurcation (case (III)). So far as we are aware, this is the first such example (in Ref. [13] a piecewise linear example is studied, for which the existence of a bifocal homoclinic bifurcation can be proved using perturbation theory, but this does not satisfy the standard smoothness conditions of Shilnikov's re- [illegible] are non-generic, having either a Hamiltonian or re- [illegible]

The observant reader will have noted that one [illegible]

$$\begin{pmatrix} \lambda_1 & 1 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \nu_1 & 1 \\ 0 & 0 & 0 & \nu_1 \end{pmatrix} \qquad (13)$$

We look at the existence of bifocal homoclinic orbits in this light elsewhere [8]: in particular, we explore several codimension two bifurcations involving bifocal homoclinic bifurcations. The normal form (13) has codimension greater than two, and we consider this to be too large for useful analysis in the absence of some concrete physical motivation.

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